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SUBSONIC LAMB WAVES IN ANISOTROPIC LAYERS†

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A six-dimensional complex formulism is developed for analysing Lamb waves propagating at subsonic velocity in anisotropic layers. An example of an elastic layer made of a material with cubic symmetry and with homogeneous boundary conditions is presented in which some of the Lamb waves, that propagate in the directions of elastic symmetry, are not present at certain values of the phase velocity. © 2001 Elsevier Science Ltd. All rights reserved.

Beginning with Lamb's paper [1], in which the basic equations for describing the propagation of elastic waves in an isotropic layer were obtained, it has been assumed in all subsequent investigations of harmonic waves with an amplitude varying throughout the thickness of a layer that the wave consists of certain partial waves (PWs) which have the form

$$\mathbf{u}_k(x) = \mathbf{m}_k e^{ir\gamma_k \mathbf{x}\cdot\nu} e^{ir(\mathbf{n}\cdot\mathbf{x} - ct)} \tag{0.1}$$

where $\mathbf{u}_k$ is the displacement field due to the k-th PW, $\mathbf{m}_k$ is the vector amplitude which, in the general case, is complex and is determined from the Christoffel equation, γ_k is a root of the Christoffel equation, r is the wave number, ν is the vector of a unit normal to the middle surface of the layer, $\mathbf{n}$ is the vector of the wave normal which lies in the middle plane and specifies the direction of propagation of the wave front and c is the phase velocity. For a Lamb wave to exist, it is necessary that all of the PWs constituting the wave have one and the same wave number and the same phase velocity.

Representation (0.1) for a PW has been used, in addition to Lamb waves, for Rayleigh waves [2] and Stoneley waves [3]. The latter are waves which propagate along a plane boundary between two different elastic half-spaces. In some investigations, such as [4, 5], for example, which are concerned with analysing Lamb waves in isotropic layers, the solution was constructed in complex Papkovich – Neuber potentials and the potentials were chosen in a form which ensured that representation (0.1) was satisfied. Representation (0.1) has also been used (see [6–10], for example) in the case of anisotropic layers for PWs which constitute a Lamb wave.

It is shown below that, in the case of Lamb waves which propagate at subsonic velocity in anisotropic layers (the phase velocity of a Lamb wave does not exceed the minimum velocity of a body wave propagating in the same direction), representation (0.1) needs correcting as, in certain cases, depending on the form of the anisotropy and the relations between the components of the elasticity tensor and the phase velocity, a Lamb wave can consist of a smaller number of PWs than was assumed in approaches dating back to Lamb's time. A proof of the existence of forbidden directions and velocities at which a Lamb wave cannot propagate serves as a result of this.

1. BASIC RELATIONS

The equations of motion of an anisotropic medium can be written in the form

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \mathrm{div}_x \mathbf{C}\cdot\cdot\nabla_x \mathbf{u} - \rho\ddot{\mathbf{u}} = 0 \tag{1.1}$$

where ρ is the density of the medium and C is a tetravalent elasticity vector. It is assumed that this tensor is positive definite

$$(\mathbf{B}\cdot\cdot\mathbf{C}\cdot\cdot\mathbf{B}) \equiv \sum_{i,j,m,n} B_{ij}C^{ijmn}B_{mn} > 0, \quad \forall_{\mathbf{B}\in\mathrm{sym}(R^3\otimes R^3),\, \mathbf{B}\neq 0}\mathbf{B} \tag{1.2}$$

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Substituting the expression for a partial wave (0.1) into the equation of motion we obtain Christoffel's equation

$$((\gamma_k \boldsymbol{\nu} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I}) \cdot \mathbf{m}_k = 0 \tag{1.3}$$

where $\mathbf{I}$ is the identity matrix. Equation (1.3) can be written in the equivalent form

$$\det((\gamma_k \boldsymbol{\nu} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I}) = 0 \tag{1.4}$$

Christoffel's equation in the form (1.4) can be considered as a polynomial equation of the sixth degree in γ_k. Since the coefficients of polynomial (1.4) are real, its complex roots occur in the set of all roots as complex conjugate pairs.

It can be shown (see [11], for example) that Eq. (1.4) does not have real roots if the phase velocity is less than the so-called lower limiting velocity $c_3^{\lim}$. In turn, $c_3^{\lim}$ does not exceed the lowest velocity of a bulk wave propagating in the same direction. Henceforth, it is assumed that the following condition is satisfied

$$c < c_3^{\lim} \tag{1.5}$$

This ensures that there are no real roots in Christoffel's equation.

2. THE SIX-DIMENSIONAL FORMULISM

Since it is not known in advance whether representation (0.1) is the only possible representation for a partial wave, we will consider the following (most general) representation for a harmonic wave with a plane wavefront and a non-uniform amplitude

$$\mathbf{v}(x'') e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)} \tag{2.1}$$

where $\mathbf{v}(x'')$ is a non-uniform, generally speaking, complex vector amplitude and $x'' = ir\boldsymbol{\nu} \cdot \mathbf{x}$ such that x'' is a dimensionless (imaginary) coordinate in the direction which is defined by the vector $\boldsymbol{\nu}$. The exponential factor in (2.1) corresponds to the motion of a plane wavefront with a phase velocity c in the direction $\mathbf{n}$. No *a priori* constraints of any kind are imposed on the smoothness of the vector amplitude $\mathbf{v}(x'')$.

Substituting representation (2.1) into the equation of motion (1.1) we obtain the second-order matrix differential equation

$$((\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \partial_{x''}^2 + (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \partial_{x''} + (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}) \mathbf{v}(x'') = 0 \tag{2.2}$$

A direct analysis of Eq. (2.2) is rather complex. The situation can be simplified if we introduce the supplementary vector function

$$\mathbf{w}(x'') = \partial_{x''} \mathbf{v}(x'') \tag{2.3}$$

Then, since the tensor $\mathbf{C}$ is positive-definite, Eq. (2.2) reduces to the first-order matrix differential equations

$$\partial_{x''} \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} \tag{2.4}$$

where

$$\mathbf{R}_6 = \begin{Vmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M} & -\mathbf{N} \end{Vmatrix}, \quad \begin{aligned} \mathbf{M} &= (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu})^{-1} \cdot (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}) \\ \mathbf{N} &= (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu})^{-1} \cdot (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \end{aligned} \tag{2.5}$$

($\mathbf{M}$ and $\mathbf{N}$ are real, third-order matrices).

When the structure of the matrix $\mathbf{R}_6$ is taken into account, its six-dimensional eigenvectors can be conveniently represented in the form

$$\mathbf{m}_6 \equiv (\mathbf{m};\ \mathbf{m}'),\quad \mathbf{m},\ \mathbf{m}' \in C^3$$

The mapping $F : C^6 \to C^3$ such that

$$F(\mathbf{m}_6) = \mathbf{m} \tag{2.6}$$

is subsequently required.

Remark 2.1. Taking account of relations (2.4) and (2.5), the determinant of the matrix can be represented in the form

$$\det \mathbf{R}_6 = (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I}) \tag{2.7}$$

The right-hand side of this equality shows that, since the tensor **C** is positive-definite, the matrix $\mathbf{R}_6$ is non-degenerate for any phase velocities with the exception of cases when $\rho c^2 = \lambda_k(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n})$ ($k = 1, 2, 3$ and λ_k denotes a characteristic number of the corresponding matrix), that is, degeneracy sets in when the phase velocity c is identical with the velocity of one of the body waves which propagate in the direction of the vector **n**. It is clear that the matrix $\mathbf{R}_6$ is non-degenerate in the case of subsonic Lamb waves.

2.2. Since the matrix $\mathbf{R}_6$ is not symmetric, its right and left eigenvectors, generally speaking, are different. Henceforth, for brevity, the term "eigenvector" is to be understood as referring to the right eigenvectors of the matrix $\mathbf{R}_6$.

2.3. In its structure, the matrix $\mathbf{R}_6$ is similar to the matrix which is used to construct the "fundamental elasticity tensor". This tensor and the corresponding matrix were previously introduced in [12] and have been used to investigate Rayleigh waves in anisotropic elastic half-spaces (see [11, 13–15], for example).

Theorem 2.1. 1. The set of roots of the characteristic polynomial of Christoffel's equation (1.4) is identical with the set of characteristic numbers of the matrix $\mathbf{R}_6$.

2. The spectral space of Christoffel's equation (1.3) is identical with the surjection (2.6) of the spectral space of the matrix $\mathbf{R}_6$.

Proof. Suppose γ_k is a root of the characteristic polynomial of Christoffel's equation (1.4) and $\mathbf{m}_k$ is the corresponding eigenvector which satisfies condition (1.3). Substituting the functions $\mathbf{v}(x'') = \mathbf{m}_k e^{\gamma_k x''}$ and $\mathbf{w}(x'') = \gamma_k \mathbf{m}_k e^{\gamma_k x''}$ corresponding to these γ_k and $\mathbf{m}_k$ into Eq. (2.4) we obtain

$$\gamma_k \begin{pmatrix} \mathbf{m}_k \\ \\ \gamma_k \mathbf{m}_k \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{m}_k \\ \\ \gamma_k \mathbf{m}_k \end{pmatrix} \tag{2.8}$$

Hence, each root of Christoffel's equation is also a characteristic number of the matrix $\mathbf{R}_6$ and the corresponding eigenvector of Christoffel's equation is identical with the vector $F(\mathbf{m}_6)$.

Now, suppose γ_k is a characteristic number and $(\mathbf{m}_k;\ \mathbf{m}'_k)$, $\mathbf{m}_k\ \mathbf{m}'_k, \in C^3$ is the corresponding eigenvector of the matrix $\mathbf{R}_6$, that is,

$$\gamma_k \begin{pmatrix} \mathbf{m}_k \\ \\ \mathbf{m}'_k \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{m}_k \\ \\ \mathbf{m}'_k \end{pmatrix} \tag{2.9}$$

Relation (2.9), together with (2.4) and (2.5), gives the equality

$$(\gamma_k^2(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + \gamma_k(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n} + \mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n} - \rho c^2\mathbf{I})\cdot\mathbf{m}_k = 0$$

which is identical with Eq. (1.3).

By virtue of what has been said at the end of Section 1, we have the following corollary.

Corollary. When condition (1.5) is satisfied, all the characteristic numbers of the matrix $\mathbf{R}_6$ are complex and occur in its spectrum as complex conjugate pairs.

3. A REPRESENTATION FOR LAMB WAVES

The structure of the general solution of the system of first-order differential equations (2.4) in C^6 is determined by the Jordan normal form of the matrix $\mathbf{R}_6$ [16]. By virtue of the corollary to Theorem 2.1, only three types of Jordan normal forms of the matrix $\mathbf{R}_6$ are possible at subsonic phase velocities which, moreover, satisfy condition (1.5)

$$\mathbf{J}_6^{(\mathrm{I})} = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_6), \quad \mathbf{J}_6^{(\mathrm{II})} = \mathrm{diag}(\Gamma_1^2, \Gamma_2^2, \gamma_3, \gamma_4), \quad \mathbf{J}_6^{(\mathrm{III})} = \mathrm{diag}(\Gamma_1^3, \Gamma_2^3) \tag{3.1}$$

$$\Gamma_j^2 = \begin{Vmatrix} \gamma_j & 1 \\ 0 & \gamma_j \end{Vmatrix}, \quad \Gamma_j^3 = \begin{Vmatrix} \gamma_j & 1 & 0 \\ 0 & \gamma_j & 1 \\ 0 & 0 & \gamma_j \end{Vmatrix}; \quad j = 1, 2$$

Remark 3.1. In expressions (3.1), it is possible to put

$$\gamma_{2k-1} = \overline{\gamma_{2k}}, \quad k = 1, 2, \ldots \tag{3.2}$$

Equalities (3.2) are possible by virtue of the corollary and Theorem 2.1. Next, since the matrix $\mathbf{R}_6$ is real, relations, similar to (3.2), are also satisfied in the case of its eigenvectors

$$(\mathbf{m}_{2k-1};\ \mathbf{m}'_{2k-1}) = (\overline{\mathbf{m}}_{2k};\ \overline{\mathbf{m}}'_{2k}) \tag{3.3}$$

Proposition 3.1. When the matrix $\mathbf{R}_6$ is normal and similar (on C) to the Jordan matrix $\mathbf{J}_6^{(\mathrm{I})}$, the following condition is satisfied: either $\gamma_k = \pm i$ or $\mathbf{m}_k \cdot \mathbf{m}_k = 0$ ($\mathbf{m}_k$ is the complex spatial image in the case of the mapping F of the corresponding six-dimensional eigenvector of the matrix $\mathbf{R}_6$).

Proof. Note that, under the conditions of Proposition 3.1, all of the eigenvectors of the matrix $\mathbf{R}_6$ can be chosen to be orthogonal (this, generally speaking, is not so in the case of matrices which are similar to $\mathbf{J}_6^{(\mathrm{II})}$ or $\mathbf{J}_6^{(\mathrm{III})}$ (([17], Chapter VII, § 5)).

Suppose $(\mathbf{m}_k;\ \mathbf{m}'_k)$, $\mathbf{m}_k\ \mathbf{m}'_k \in C^3$ is an eigenvector of the matrix $\mathbf{R}_6$ which corresponds to the characteristic number γ_k. By virtue of relation (3.3), the condition for the eigenvectors $(\mathbf{m}_{2k-1};\ \mathbf{m}'_{2k-1})$ and $(\mathbf{m}_{2k};\ \mathbf{m}'_{2r})$ to be orthogonal has the form

$$\mathbf{m}_{2k-1} \cdot \mathbf{m}_{2k-1} = -\mathbf{m}'_{2k-1} \cdot \mathbf{m}'_{2k-1} \tag{3.4}$$

It is obvious that a similar relation also holds for vectors with even subscripts. Next,

$$\gamma_k \mathbf{m}_k = \mathbf{m}'_k, \quad \forall k \tag{3.5}$$

follows from relation (2.9) when account is taken expression (2.4), which specifies the structure of the matrix $\mathbf{R}_6$.

Relations (3.4) and (3.5) give

$$(1 + \gamma_k^2)\mathbf{m}_k \cdot \mathbf{m}_k = 0 \tag{3.6}$$

which completes the proof.

The transition from first-order system (2.4) to initial system (2.2), taking account of the results of the general theory of matrix differential equations ([16], Chapter 4), enables one to represent the corresponding general solutions (the matrix of the fundamental solutions) in the form

$$\begin{aligned}
\mathbf{v}^{(\mathrm{I})}(x'') &= \sum_{k=1}^{6} C_k \mathbf{m}_k e^{\gamma_k x''} \\
\mathbf{v}^{(\mathrm{II})}(x'') &= \sum_{k=1}^{2} (C_{2k-1} + C_{2k} x'') \mathbf{m}_k e^{\gamma_k x''} + \sum_{k=3}^{4} C_{k+2} \mathbf{m}_k e^{\gamma_k x''} \\
\mathbf{v}^{(\mathrm{III})}(x'') &= \sum_{k=1}^{2} (C_{3k-2} + C_{3k-1} x'' + C_{3k} x''^2) \mathbf{m}_k e^{\gamma_k x''}
\end{aligned} \tag{3.7}$$

where C_k are complex coefficients to be determined.

Remark 3.2. In the case of scalar differential equations, the unknown coefficients C_k are independent and are determined, as a rule, from the boundary conditions. In the case of matrix differential equations similar to (3.4), these coefficients may be dependent ([16], Chapter 4) and the possible relations between them are determined by substituting the corresponding solution into the initial equation.

Proposition 4.2. 1. In the representation for $\mathbf{v}^{(I)}$, all the coefficients C_k are independent.
2. In the representation for $\mathbf{v}^{(II)}$, the coefficients C_2 and C_4 are zero.
3. In the representation for $\mathbf{v}^{(III)}$, the coefficients C_2, C_3 and C_5, C_6 are equal to zero.

Proof. Assertion 1 is obvious since the matrix $\mathbf{R}_6$ does not contain Jordan blocks.

In order to prove assertion 2, we substitute the solution corresponding to one of the Jordan blocks (for example, the block containing γ_1) into (2.4). This gives

$$(C_1\mathbf{A}+xC_2\mathbf{A}+C_2\mathbf{B})\cdot\mathbf{m}_1e^{\gamma_1x''}=0$$

$$\mathbf{A}=\gamma_1^2\mathbf{I}+\gamma_1\mathbf{N}+\mathbf{M},\quad \mathbf{B}=2\gamma_1\mathbf{I}+\mathbf{N} \tag{3.8}$$

The matrices $\mathbf{M}$ and $\mathbf{N}$ are defined by formulae (2.5). It follows from (3.8) that

$$\mathbf{m}_1\in\ker\mathbf{A} \tag{3.9}$$

and either $C_2 = 0$ or the following condition is satisfied

$$\mathbf{m}_1\in\ker\mathbf{B} \tag{3.10}$$

We will now show that it is impossible to satisfy the last two relations in the case of complex γ_1. When account is taken of (2.5), condition (2.9) gives

$$\overline{\mathbf{m}}_1\cdot(2\gamma_1(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})+(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n}+\mathbf{n}\cdot\mathbf{C}\cdot\boldsymbol{\nu}))\cdot\mathbf{m}_1=0 \tag{3.11}$$

However, when the fact that the tensor $(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})$ is positive-definite is taken into account, condition (3.11) is incompatible with the condition $\mathrm{Im}(\gamma_1)\neq 0$.

In order to prove assertion 3°, we substitute the solution corresponding to the Jordan block containing γ_1 into Eq. (2.4). This gives

$$((C_1+xC_2+x^2C_3)\mathbf{A}+(C_2+2xC_3)\mathbf{B}+2C_3\mathbf{I})\cdot\mathbf{m}_1e^{\gamma_1x''}=0$$

Further analysis shows that condition (3.9) is satisfied and assertion 3° follows from this when account is taken of equality (3.11).

By virtue of Proposition 3.2, the structure of a Lamb wave depends on the existence and rank of Jordan blocks in the Jordan normal form of the matrix $\mathbf{R}_6$. Taking account of relations (3.7), we can write the representation for a Lamb wave in the form

$$\mathbf{u}(\mathbf{x})=\sum_{k=1}^{p}C_k\mathbf{m}_ke^{ir\gamma_k\mathbf{x}\cdot\boldsymbol{\nu}}e^{ir(\mathbf{n}\cdot\mathbf{x}-ct)},\quad p=\begin{cases}6 & \text{for } \mathbf{J}_6^{(I)}\\ 4 & \text{for } \mathbf{J}_6^{(II)}\\ 2 & \text{for } \mathbf{J}_6^{(III)}\end{cases} \tag{3.12}$$

In (3.12), the arbitrary coefficients C_k, corresponding to $\mathbf{J}_6^{(II)}$, and $\mathbf{J}_6^{(III)}$, have now already been renumbered compared with the numbering in relations (3.7).

4. THE DISPERSION EQUATION

It is assumed that the homogeneous boundary conditions in the stresses

$$\mathbf{x}\cdot\boldsymbol{\nu}=\pm h:\quad \mathbf{t}\equiv\pm\boldsymbol{\nu}\cdot\mathbf{C}\cdot\cdot\nabla_{\mathbf{x}}\mathbf{u}=0 \tag{4.1}$$

are satisfied on the surfaces of a layer ($2h$ is the layer thickness).

Substituting the displacement field (3.12) into boundary conditions (4.1) and changing to the dimensionless coordinates $x''=ir(\boldsymbol{\nu}\cdot\mathbf{x})$ we obtain

$$\sum_{k=1}^{p} C_k \mathbf{t}_k e^{\pm\gamma_k\xi} = 0$$
$$\mathbf{t}_k = (\gamma_k \boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu} + \boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n})\cdot\mathbf{m}_k, \quad \xi = irh \tag{4.2}$$

(the parameter p is defined by the group of equalities in (3.12)).

The boundary conditions in the form (4.2) can be treated as a non-trivial solution of system (4.2) with respect to the unknown coefficients C_k ($k = 1, \dots, p$); the latter is equivalent to all of the complex determinants of order p vanishing:

$$\delta_p \equiv \det_p \begin{Vmatrix} \mathbf{t}_1 e^{+\gamma_1\xi} \dots \mathbf{t}_p e^{+\gamma_p\xi} \\ \mathbf{t}_1 e^{-\gamma_1\xi} \dots \mathbf{t}_p e^{-\gamma_p\xi} \end{Vmatrix} = 0 \tag{4.3}$$

Equations (4.3) are the required dispersion relations which give, by mean of the preliminary solution of differential equation (2.4), the link between the phase velocity c and the wave number r or the phase velocity and the angular frequency $\omega = rc$.

The question of the solvability of Eqs. (4.3) in the case of arbitrary anisotropy was not investigated but the following assertions hold.

Proposition 4.1. Suppose $\mathrm{Re}(\gamma_k) = 0$ for any k. Then, a Lamb wave with a phase velocity which satisfies condition (1.5) cannot just consist of the corresponding partial wave.

Proof. Suppose the corresponding partial wave satisfies Eqs. (4.3). This is equivalent to the equality

$$\mathbf{t}_k = 0 \tag{4.4}$$

On multiplying both sides of this equality by the vector $\overline{\mathbf{m}}_k$, we obtain

$$\gamma_k \boldsymbol{\nu}\otimes\overline{\mathbf{m}}_k \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{m}_k \otimes \boldsymbol{\nu} + \boldsymbol{\nu}\otimes\overline{\mathbf{m}}_k \cdot\cdot \mathbf{C} \cdot\cdot \mathbf{m}_k \otimes \mathbf{n} = 0 \tag{4.5}$$

It now remains to note that, when $\mathrm{Im}\,(\gamma_k) \neq 0$, the imaginary part of the expression on the left-hand side of (4.5) also non-zero by virtue of the fact that the elasticity tensor is positive-definite. Hence, it is not possible for the left-hand side of this equality to be equal to zero.

Proposition 4.2. Under the conditions of Proposition 4.1, a Lamb wave cannot be formed by two partial waves corresponding to conjugate characteristic numbers (to roots of the characteristic polynomial).

Proof. Suppose a non-trivial solution of system (4.2) exists for any conjugate characteristic numbers, such as γ_1 and γ_1, for example. In this case, we must put $p = 2$, and conditions (4.3) that the second order determinants should vanish reduce to two equations. The first equation can be represented in the form

$$\mathbf{t}_1 \times \overline{\mathbf{t}}_1 = 0 \tag{4.6}$$

where $\mathrm{Im}\,(\gamma_1) \neq 0$ by virtue of (1.5) so that $\mathrm{Im}\,(\mathbf{t}_1) \neq 0$. Equation (5.6) expresses the condition for the collinearity of the vectors $\mathbf{t}_1$ and $\mathbf{t}_1$:

$$\mathbf{t}_1 = \alpha\mathbf{e}, \quad \overline{\mathbf{t}}_1 = \overline{\alpha}\mathbf{e} \tag{4.7}$$

where α is a certain non-zero complex constant and $\mathbf{e} \in R^3$. When account is taken of condition (4.7), we write the second equation following from (4.3) in the form

$$\delta_2 \equiv \det_2 \begin{Vmatrix} \alpha e^{+\gamma_1\xi} & \overline{\alpha} e^{+\overline{\gamma}_1\xi} \\ \alpha e^{-\gamma_1\xi} & \overline{\alpha} e^{-\overline{\gamma}_p\xi} \end{Vmatrix} = 0 \tag{4.8}$$

It is clear that, when $\mathrm{Im}\,(\gamma_1) \neq 0$ and $\alpha \neq 0$, it is impossible for the left-hand side of equality (5.8) to vanish.

Corollary. Lamb waves corresponding to the normal Jordan form $\mathbf{J}_6^{(\mathrm{III})}$ do not exist under the conditions of Proposition 4.2.

5. LAYERS OF MATERIAL WITH CUBIC SYMMETRY

It is shown below that the Jordan blocks (of rank 2) in the Jordan normal form of the matrix $\mathbf{R}_6$ occur even in layers made of materials with cubic symmetry when the middle plane coincides with a plane of elastic symmetry and the directions of propagation coincide with one of the crystallographic axes. Suppose the elasticity tensor of a crystal with cubic symmetry in the crystallographic axes specified by the vectors $\mathbf{e}_k$ ($k = 1, 2, 3$) has the form

$$\begin{aligned} \mathbf{C} &= \eta \sum_k \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_k + \lambda \sum_{k \neq m} \mathbf{e}_k \otimes \mathbf{e}_k \otimes \mathbf{e}_m \otimes \mathbf{e}_m + \\ &+ 4\mu \sum_{k<m} \operatorname{sym}(\mathbf{e}_k \otimes \mathbf{e}_m) \otimes \operatorname{sym}(\mathbf{e}_k \otimes \mathbf{e}_m) \\ &\operatorname{sym}(\mathbf{e}_k \otimes \mathbf{e}_m) = \tfrac{1}{2}(\mathbf{e}_k \otimes \mathbf{e}_m + \mathbf{e}_m \otimes \mathbf{e}_k) \end{aligned} \tag{5.1}$$

(η, λ, μ) are the constants of elasticity).

The condition of positive definiteness (1.2), which is imposed on the tensor $\mathbf{C}$, gives

$$\eta - \lambda > 0, \quad \eta + 2\lambda > 0, \quad \mu > 0 \tag{5.2}$$

Suppose the unit vectors $\boldsymbol{\nu}$, the normals to the middle plane of the layer, and $\mathbf{n}$, the directions of propagation of a Lamb wave, are orientated along the crystallographic axes of the crystal. Substituting the elasticity tensor (5.1) into the last two equalities of (2.5) are obtain

$$\begin{aligned} \mathbf{M} &= \left(\frac{\mu - \rho c^2}{\eta} \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \frac{\eta - \rho c^2}{\mu} \mathbf{n} \otimes \mathbf{n} + \frac{\mu - \rho c^2}{\mu} \mathbf{w} \otimes \mathbf{w} \right), \quad \mathbf{w} = \boldsymbol{\nu} \times \mathbf{n} \\ \mathbf{N} &= \left(\frac{\lambda + \mu}{\eta} \boldsymbol{\nu} \otimes \mathbf{n} + \frac{\lambda + \mu}{\mu} \mathbf{n} \otimes \boldsymbol{\nu} \right) \end{aligned} \tag{5.3}$$

When account is taken of equalities (5.3), analysis of the structure of the matrix $\mathbf{R}_6$ leads to the conclusion that the following assertion holds.

Proposition 5.1. 1. The relation between the phase velocity, the density and the constants of elasticity of a cubic crystal

$$\rho c^2 = 2 \frac{|\lambda + \mu|}{(\eta - \mu)^2} \sqrt{\eta\mu(\eta + \lambda)(\lambda + 2\mu - \eta)} - \frac{(\eta + \mu)(\eta + \lambda)(\lambda + 2\mu - \eta)}{(\eta - \mu)^2} \tag{5.4}$$

is necessary and sufficient of the occurrence of the Jordan normal form $\mathbf{J}_6^{(\mathrm{II})}$.

2. The characteristic numbers of the matrix $\mathbf{R}_6$, which correspond to (5.4) can be represented in the form

$$\begin{aligned} &\gamma_1 = i\psi_\mu^{1/4}\psi_\eta^{1/4}, \quad \gamma_2 = -\gamma_1, \quad \gamma_3 = i\psi_\mu^{1/2}, \quad \gamma_4 = -\gamma_3 \\ &\psi_\mu = 1 - \rho c^2/\mu, \quad \psi_\eta = 1 - \rho c^2/\eta \end{aligned} \tag{5.5}$$

where γ_1 and γ_2 correspond to Jordan blocks.

3. The amplitudes $\mathbf{m}_k$, corresponding to the characteristic numbers γ_1, have the form (p is a normalizing factor)

$$\begin{aligned} &\mathbf{m}_1 = p(i\boldsymbol{\nu}\psi_\eta^{1/4} + \mathbf{n}\psi_\mu^{1/4}), \quad \mathbf{m}_2 = \overline{\mathbf{m}}_1, \quad \mathbf{m}_3 = \mathbf{m}_4 = \mathbf{w} \\ &p = (\psi_\mu^{1/2} + \psi_\eta^{1/2})^{-1/2} \end{aligned} \tag{5.6}$$

By substituting expressions (5.5) and (5.6) into dispersion equation (4.3) and taking account of equality (5.1) it can be shown that Eq. (4.3) is not satisfied by any values of the parameter ξ. In fact, substitution of the amplitudes (5.6) into the dispersion equation leads, after some reduction, to two independent equations with (6×2)-matrices, which have already been encountered in the proof of Proposition (4.2).

We therefore obtain the following theorem on the existence of forbidden directions and velocities for Lamb waves.

Theorem 5.1. Under the conditions of Proposition (5.1), Lamb waves, propagating in the direction of the crystallographic axis of a cubic crystal, do not exist.

Remark 5.1. In the formulation of Theorem 5.1, we speak of the impossibility of the propagation of a Lamb wave at a completely defined value of the phase velocity which satisfies condition (5.4). At other values of the phase velocity, when condition (5.4) is violated, Lamb waves can perfectly well exist.

From a practical point of view, it is interesting that the disappearance of Lamb waves at certain phase velocities is observed for all cubic crystals for which the components of the elasticity tensor satisfy the condition

$$\lambda + 2\mu > \eta$$

Actually, when this condition and the condition of positive definiteness (5.2) are satisfied, the phase velocity, defined according to (5.4), is positive and satisfies condition (1.5).

A simple physical interpretation can be given of the disappearance of Lamb waves which propagate in the directions of elastic symmetry of cubic crystals if it is noted that, for fixed values of the constants of elasticity, density and values of the phase velocity which differ only slightly from those determined using (5.4), the Jordan normal form of the matrix $\mathbf{R}_6$ now does not contain Jordan blocks and a Lamb wave is formed by six partial waves. In this case, dispersion equation (4.3) splits into an equation containing a fourth-order determinant and an equation with a second-order determinant, which is analogous to Eq. (4.8). It has been established in Proposition 4.2 that this last equation does not have non-trivial solutions. A more detailed treatment of the first equation shows that it has non-trivial solutions everywhere except at the point c_0, where the phase velocity satisfies condition (5.4). However, in the neighbourhood of this point when $c \to c_0$, the characteristic numbers of the matrix $\mathbf{R}_6$ and the corresponding amplitudes satisfy the conditions

$$\begin{gathered} \gamma_3 \to \gamma_1, \quad \gamma_4 \to \gamma_2 \\ \mathbf{m}_3 \to \mathbf{m}_1, \quad \mathbf{m}_4 \to \mathbf{m}_1 \end{gathered} \tag{5.7}$$

Here, in accordance with the notation of (3.2), $\gamma_2 = \bar{\gamma}_1$ and $\gamma_4 = \bar{\gamma}_3$, and similar equalities hold in the case of the amplitudes. At the same time, the corresponding root solutions of Eq. (4.2) when $c \to c_0$ satisfy the conditions

$$C_3 \to -C_1, \quad C_4 \to -C_2 \tag{5.8}$$

By combining relations (5.7) and (5.8) it can be shown that, when $c \to c_0$, the partial waves, multiplied by the corresponding coefficients, are mutually annihilated, which also leads to the disappearance of the resulting Lamb wave.

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